Finitary Power Monoids: Atomicity, Divisibility, and Beyond

Jiya Dani, Leo Hong, and Shimon Schlessinger

MIT PRIMES-USA

(Mentored by Dr. Felix Gotti and Benjamin Li)

Fall-Term PRIMES Conference

October 12, 2024

Outline

- Background
- Weaker Notions of Atomicity
- Weaker Notions of ACCP
- Further Divisibility Properties

Monoids

Definition: A monoid is a pair $(M, +)$, where M is a set and $+$ is a binary operation satisfying the following conditions:

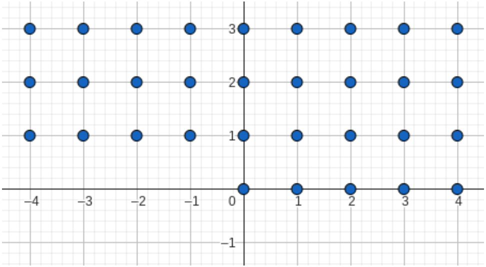
- $+$ is associative: $a + (b + c) = (a + b) + c$ for all $a, b, c \in M$,
- there exists $0 \in M$ such that $0 + m = m$ for all $m \in M$,
- $+$ is commutative: $a + b = b + a$ for all $a, b \in M$.

Submonoids

Definition: A subset N of a monoid M is called a submonoid of M if $0 \in N$ and N is closed under $+$.

Examples:

- Puiseux monoids are submonoids of $\mathbb{Q}_{\geq 0}$ (by definition).
- The monoid $(\mathbb{N}_0 \times \{0\}) \cup (\mathbb{Z} \times \mathbb{N})$ is a submonoid of $\mathbb{Z}^2$.



Monoids (cont.)

Definitions:

1. For a subset S of a monoid M, the submonoid of M generated by S is defined as

$$\langle S \rangle := \Big\{ \sum_{s \in S} n_s s : n_s \in \mathbb{N}_0 \text{ with } n_s \neq 0 \text{ for only finitely many } s \in S \Big\}.$$

2. For a monoid M and $b, c \in M$, we say b divides c (and write $b \mid_M c$) if $a + b = c$ for some $a \in M$.

Examples:

- Every submonoid of $\mathbb{N}_0$ can be generated by a finite set.
- The monoid generated by the infinite set $\{\frac{1}{2^i} : i \in \mathbb{N}\}$ is the Puiseux monoid of non-negative dyadic rationals $\mathbb{Z}\big[\frac{1}{2}\big]_{\geq 0} = \big\{\frac{n}{2^i} : i, n \in \mathbb{N}_0\big\}$, which cannot be generated by a finite set.

Finitary Power Monoids

Definition: The finitary power monoid of a monoid M is the set of non-empty finite subsets of M, denoted by $\mathcal{P}_{\text{fin}}(M)$, with the so-called sumset as its binary operation: for $S, T \in \mathcal{P}_{\text{fin}}(M)$,

$$S + T := \{s + t : s \in S, t \in T\}.$$

Example: In $\mathcal{P}_{\text{fin}}(\mathbb{N}_0)$,

- $\{0,1\} + \{0,1\} = \{0,1,2\}$ and
- $\{0,1\} + \{0,1,2\} = \{0,1\} + \{0,2\} = \{0,1,2,3\}$.

Remark. For cancellative monoids M, the power monoid $\mathcal{P}_{\text{fin}}(M)$ is not necessarily cancellative.

Linearly Orderable Monoids

Definitions: Let $(M, \leq)$ be a monoid with a total order relation $\leq$.

1. $(M, \leq)$ is a linearly orderable monoid if $a \leq b$ implies $a + c \leq b + c$ for all $a, b, c \in M$.

2. A linearly orderable monoid $(M, \leq)$ is a positive monoid if $0 \leq m$ for all $m \in M$.

3. A positive monoid $(M, \leq)$ is Archimedean if for all non-zero $a, b \in M$, $na > b$ for some $n \in \mathbb{N}$.

Example: Puiseux monoids are Archimedean under the standard order relation.

Atomicity

Definitions: Let M be a monoid.

1. Invertible elements are called units. The group of units is denoted by $\mathcal{U}(M)$. If $a = b + u$ where u is a unit, then a and b are called associates. If $\mathcal{U}(M) = \{0\}$, then M is called reduced.
2. $a \in M \backslash \mathcal{U}(M)$ is an atom if $a = b + c$ implies b or c is a unit. The set of atoms is denoted by $\mathcal{A}(M)$. If $\mathcal{A}(M) = \emptyset$, then M is called antimatter.
3. M is called atomic if every $b \in M$ is an atomic element, that is, b can be written as a finite sum of atoms and units.

Examples:

- The (dyadic) positive monoid $\mathbb{Z}\left[\frac{1}{2}\right]_{\geq 0}$ generated by $\{\frac{1}{2^i} : i \in \mathbb{N}\}$ is antimatter, reduced, and not atomic.
- The positive monoid generated by $\left\{\frac{1}{p} : p \in \mathbb{P}\right\}$ is atomic with set of atoms $\left\{\frac{1}{p} : p \in \mathbb{P}\right\}$.

Atomic Monoid Example

Example: We claim that the positive monoid M generated by $\{\frac{1}{p} : p \in \mathbb{P}\}$ is atomic with the set of atoms being $\{\frac{1}{p} : p \in \mathbb{P}\}$.

1. First, note that $\mathcal{A}(M) \subseteq \{\frac{1}{p} : p \in \mathbb{P}\}$ as $\{\frac{1}{p} : p \in \mathbb{P}\}$ generates M.
2. It suffices to show that for any prime $q \in \mathbb{P}$,

$$\frac{1}{q} \notin \left\langle \frac{1}{p} : p \in \mathbb{P} \setminus \{q\} \right\rangle.$$

3. Note that a linear combination of the reciprocals of a set of integers has a denominator which divides the product of the integers.
4. But no product of a set primes is divisible by a prime not in the set, proving the statement in (2).

Maximal Common Divisors (MCD)

Definitions: Let M be a monoid, and let S be a non-empty finite subset of M.

1. An element $m \in M$ is a maximal common divisor (MCD) of S if m is a common divisor of S and no other common divisor $d \in M$ of S exists such that $m \mid_M d$ but $d \nmid_M m$.
2. For each $k \in \mathbb{N}$, the monoid M is called a k-MCD monoid if every subset of M with size k has an MCD in M. In addition, M is called an MCD monoid if it is k-MCD for all $k \in \mathbb{N}$.

Examples:

- Every finitely generated monoid is an MCD monoid.
- The monoid generated by $\left\{\frac{1}{2^i}, \frac{1}{3} + \frac{1}{2^i} : i \in \mathbb{N}\right\}$ is not a 2-MCD monoid as its subset $\left\{1, \frac{4}{3}\right\}$ does not have an MCD.

Non-MCD Monoid Example

Example: We claim that the monoid $M = \left\langle \frac{1}{2^i}, \frac{1}{3} + \frac{1}{2^i} : i \in \mathbb{N} \right\rangle$ is not 2-MCD, as $\{1, \frac{4}{3}\}$ does not have an MCD.

1. By inspection, every divisor of 1 is of the form $\frac{n}{2^k} \leq 1$ for $n, k \in \mathbb{N}_0$.
2. Furthermore, every such element, excluding 1, is a common divisor of $\{1, \frac{4}{3}\}$.
3. Note that $1 \nmid_M \frac{4}{3}$, so the set of common divisors of $\{1, \frac{4}{3}\}$ are the dyadic rationals less than 1.
4. Therefore, for any common divisor $d = \frac{n}{2^k}$, the rational $d_0 = d + \frac{1}{2^{k+1}} \in M$ is another common divisor of $\{1, \frac{4}{3}\}$ and thus d is not an MCD.
5. Hence, $\{1, \frac{4}{3}\}$ does not have an MCD in M.

MCD and Atomicity

Theorem (D.-Gotti-H.-Li-S, 2024)

Let M be a monoid, and let $k \in \mathbb{N}$. Then the following are equivalent:

- M is an MCD monoid.
- $\mathcal{P}_{\text{fin}}(M)$ is an MCD monoid.
- $\mathcal{P}_{\text{fin}}(M)$ is a k-MCD monoid.

Moreover, if M is atomic monoid, then $\mathcal{P}_{\text{fin}}(M)$ is atomic if and only if M is an MCD monoid.

Theorem (Gonzalez-Li-Rabinovitz-Rodriguez-Tirador, 2023), (D.-Gotti-H.-Li-S, 2024)

There exists an atomic Puiseux monoid M such that $\mathcal{P}_{\text{fin}}(M)$ is not atomic.

Near Atomicity

Definition: A linearly orderable monoid M is nearly atomic if there exists $a \in M$ such that $a + b$ is atomic for every $b \in M$.

Remarks:

- For a nearly atomic monoid M, the element $a \in M$ mentioned before must be atomic.
- Atomicity $\implies$ near atomicity.

Example: Let α be irrational and $\phi : \mathbb{Q}_{\geq 0} \to \mathbb{P}$ be an injective mapping. We claim that the monoid $M = \left\langle q, \frac{q+\alpha}{\phi(q)} : q \in \mathbb{Q}_{\geq 0} \right\rangle$ is nearly atomic but not atomic:

- None of the rationals are atomic in M.
- $\frac{q+\alpha}{\phi(q)}$ is an atom for $q \in \mathbb{Q}_{\geq 0}$, so $\alpha + m$ is atomic for all $m \in M$.

Theorem (D.-Gotti-H.-Li-S, 2024)

There exists an atomic Puiseux monoid M such that $\mathcal{P}_{\text{fin}}(M)$ is not nearly atomic.

Almost Atomicity and Quasi-Atomicity

Definitions: Let M be a monoid.

- M is almost atomic if for every $b \in M$, there exists an atomic element $a \in M$ such that $a + b$ is atomic.
- M is quasi-atomic if for every $b \in M$, there exists an element $a \in M$ such that $a + b$ is atomic.

Remark: near atomicity $\implies$ almost atomicity $\implies$ quasi-atomicity.

Example: The monoid $M = \left\langle \frac{1}{2^i}, \frac{1}{3^i} : i \in \mathbb{N} \right\rangle_{\geq \frac{4}{3}}$ is quasi-atomic but not almost atomic.

- The only atom is $\frac{4}{3}$, however no multiple of $\frac{4}{3}$ added to $\frac{1}{2}$ is atomic.
- Every element divides some multiple of $\frac{4}{3}$.

Theorem (D.-Gotti-H.-Li-S, 2024)

There exists an almost atomic monoid M such that $\mathcal{P}_{\text{fin}}(M)$ is not quasi-atomic.

ACCP

Definitions: Let M be a monoid, and let I be a subset of M.

1. The set I is an ideal of M if $I + M \subseteq I$. The ideal I is principal if $I = x + M$ for some $x \in M$.
2. The monoid M satisfies the ACCP (ascending chains condition on principal ideals) if every ascending chain of principle ideals

$$b_1 + M \subseteq b_2 + M \subseteq \cdots$$

eventually stabilizes: for some $N \in \mathbb{N}$, the equality $b_m + M = b_n + M$ holds for all $m, n > N$).

Almost ACCP and Quasi-ACCP

Definitions: Let M be a monoid.

- For an element $s \in M$, we say s satisfies the ACCP if every ascending chain of principal ideals starting from $s + M$ stabilizes.
- M is almost ACCP if for every non-empty subset S of M, there exists an atomic common divisor d of S such that $s - d$ satisfies the ACCP for some $s \in S$.
- M is quasi-ACCP if for every subset S of M, there exists a common divisor d of S such that $s - d$ satisfies the ACCP for some $s \in S$.

Remark: ACCP $\implies$ Almost ACCP $\iff$ quasi-ACCP and atomic.

Examples:

- The monoid $\mathbb{Q}_{\geq 0}$ is quasi-ACCP but not almost ACCP.
- For any rational $q \in (0, 1)$ such that $\frac{1}{q} \notin \mathbb{Z}$, the monoid generated by $\{q^n : n \in \mathbb{N}\}$ is almost ACCP but not ACCP.

Ascent of Notions Weaker than the ACCP

Theorem (Gonzalez-Li-Rabinovitz-Rodriguez-Tirador, 2023):
If a monoid M is ACCP, then $\mathcal{P}_{\text{fin}}(M)$ is ACCP.

Theorem (D.-Gotti-H.-Li-S, 2024)

Let M be a linearly orderable monoid.

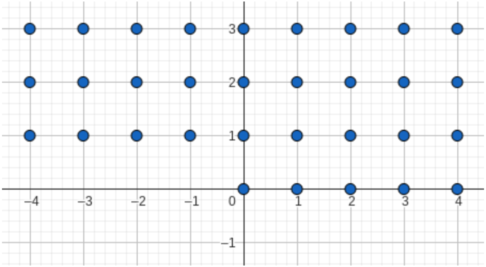
1. If a monoid M is almost ACCP, then $\mathcal{P}_{\text{fin}}(M)$ is almost ACCP.
2. If a monoid M is quasi-ACCP, then $\mathcal{P}_{\text{fin}}(M)$ is quasi-ACCP.

Furstenberg Property

Definition: A monoid M satisfies the Furstenberg property if every non-unit is divisible by at least one atom.

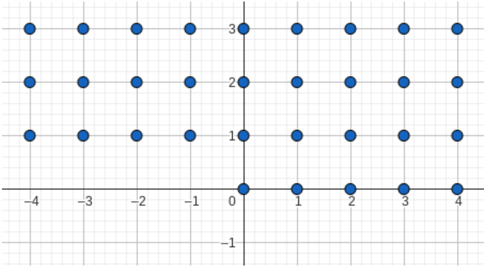
Examples:

- Every atomic monoid satisfies the Furstenberg property.
- The monoid $M = (\mathbb{N}_0 \times \{0\}) \cup (\mathbb{Z} \times \mathbb{N})$ satisfies the Furstenberg property; however, it is not atomic.



A Furstenburg non-Atomic Monoid

Example: We claim that the monoid $M = (\mathbb{N}_0 \times \{0\}) \cup (\mathbb{Z} \times \mathbb{N})$ satisfies the Furstenberg property; however, it is not atomic.



- By inspection, $(1, 0)$ is an atom.
- Any nonzero element is divisible by $(1, 0)$, so M is Furstenburg and $\mathcal{A}(M) = \{(1, 0)\}$.
- $\langle\{\mathcal{A}(M)\}\rangle = \mathbb{N}_0 \times \{0\} \neq M$, so M is not atomic.

Nearly Furstenberg Property

Definition: A monoid M satisfies the nearly Furstenberg property if there exists an element $c \in M$ such that for every non-unit $b \in M$, there exists an atom a such that $a \mid_M b + c$ but $a \nmid_M c$.

Remark: Furstenberg $\Longrightarrow$ nearly Furstenberg.

Almost Furstenberg Property

Definition: A monoid M satisfies the almost Furstenberg property if for every non-unit $b \in M$, there exists an atomic element c and an atom a such that $a \mid_M b + c$ but $a \nmid_M c$.

Remark: Furstenberg $\Longrightarrow$ almost Furstenberg.

Theorem (Lin-Rabinovitz-Zhang 2023)
There exist infinitely many non-isomorphic Puiseux monoids which satisfy the following properties:

- nearly Furstenberg but not almost Furstenberg;
- almost Furstenberg but not nearly Furstenberg;
- almost Furstenberg and nearly Furstenberg but not Furstenberg.

Quasi-Furstenberg Property

Definition: A monoid M satisfies the quasi-Furstenberg property if for each non-unit $b \in M$, there exists some $c \in M$ and atom $a \in M$ such that $a \mid_M b + c$ but $a \nmid_M c$.

Remarks:

- Nearly Furstenberg $\Longrightarrow$ quasi-Furstenberg.
- Almost Furstenberg $\Longrightarrow$ quasi-Furstenberg.
- (Lin-Rabinovitz-Zhang 2023) A Puiseux monoid M satisfies the quasi-Furstenberg property $\iff$ M is quasi-atomic $\iff$ M is not antimatter.

Example: The monoid M generated by $\{\frac{1}{2}\} \cup \{\frac{1}{3^i} : i \in \mathbb{N}\}$ satisfies the quasi-Furstenberg property, however does not satisfy either of the nearly Furstenberg and almost Furstenberg properties.

- $\frac{1}{2}$ is an atom, and every element divides a multiple of $\frac{1}{2}$.

Ascent of Notions Weaker than the Furstenberg Property

Theorem (D.-Gotti-H.-Li-S, 2024)

Let M be a linearly orderable monoid. For the following properties P, if M satisfies P, then $\mathcal{P}_{\text{fin}}(M)$ also satisfies P.

1. Furstenberg property
2. Nearly Furstenberg property
3. Almost Furstenberg property
4. Quasi-Furstenberg property

Divisibility and Irreducibility: IDF and TIDF

Definitions: A monoid is IDF (irreducible divisor finite) if every element is divisible by finitely many atoms up to associate. A monoid is TIDF (tight irreducible divisor finite) if it is IDF and satisfies the Furstenberg property.

Examples:

- Every antimatter monoid is IDF.
- Every finitely generated monoid is TIDF.
- The monoid generated by $\left\{\frac{1}{p} : p \in \mathbb{P}\right\}$ is not IDF ($\frac{1}{p} \mid_M 1$ for every prime p). However, it is atomic.

Theorem (D.-Gotti-H.-Li-S, 2024)

If M is a positive Archimedean TIDF monoid, then $\mathcal{P}_{\text{fin}}(M)$ is TIDF.

Theorem (D.-Gotti-H.-Li-S, 2024)

There exists a positive TIDF monoid M such that $\mathcal{P}_{\text{fin}}(M)$ is not IDF.

Acknowledgments

- We are grateful to our PRIMES mentors, Dr. Felix Gotti and Benjamin Li, for introducing us to the field of additive combinatorics and also for supporting us in conducting research in the same field.
- Finally, we thank the organizers of the PRIMES program for bringing us together and making this research opportunity possible.

References

- J. G. Boynton and J. Coykendall, *On the graph divisibility of an integral domain*, Canad. Math. Bull. **58** (2015) 449–458.
- Y. Fan and S. Tringali, *Power monoids: A bridge between factorization theory and arithmetic combinatorics*, J. Algebra **512** (2018) 252–294.
- N. Lebowitz-Lockard, *On domains with properties weaker than atomicity*, Comm. Algebra **47** (2019) 1862–1868.
- F. W. Levi, *Arithmetische Gesetze im Gebiete diskreter Gruppen*, Rend. Circ. Mat. Palermo **35** (1913) 225–236.
- F. Gotti, J. Vulakh, *On the atomic structure of torsion-free monoids*, Semigroup Forum **107** (2023) 1-22.
- F. Gotti and M. Zafrullah, *Integral domains and the IDF property*, J. Algebra **614** (2023) 564–591.

References

V. Gonzalez, B. Li, H. Rabinovitz, P. Rodriguez, M. Tirador, *On the atomicity of power monoids of Puiseux monoids*, International Journal of Algebra and Computation (to appear). Preprint on arXiv: https://arxiv.org/abs/2401.12444

A. Grams and H. Warner, *Irreducible divisors in domains of finite character*, Duke Math. J. **42** (1975) 271–284.

A. Lin, H. Rabinovitz, and Q. Zhang, *The Furstenberg property in Puiseux monoids*. Submitted. Preprint on arXiv: https://arxiv.org/abs/2309.12372

M. Roitman, *Polynomial extensions of atomic domains*, J. Pure Appl. Algebra **87** (1993) 187–199.

End of Presentation

THANK YOU!